C455 Algorithms Analysis Recurrence Relations

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Definitions

- A **recurrence relation** for a sequence F(n), $n \in N$, is an equation of the form
- $F(n)=\Phi(n, F(n-1), F(n-2),...F(0))$
- A solution for the recurrence relation is a formula for F(n) that only depends on n.
- A **linear** recurrence relation is of the form
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) +...c_n(n)F(0)+\phi(n)$
- If $\phi(n)$ is missing, then the relation is called **homogeneous**.

Definitions

- A **first order** recurrence relation depends only on n and F(n-1).
- A **second order** relation depends on n, F(n-1), F(n-2). Third order is defined similarly and so on...
- For a recurrence relation of first order we must know F(0) (or F(1) or wherever the sequence starts from).
- For a recurrence relation of second order we need to know F(0) and F(1) and so on.

Computational Efficiency

- F(1)=1, F(2)=2, F(n)=F(n-2)-nF(n-1)

```
long F(long n)
{
  if (n<=2)
    return n;
  else
    return F(n-2)-n*F(n-1);
}
// Exponential complexity.
// Lots of repetition
```

```
long F(long n)
{
  if (n <= 2)
    return n;
  long F_n=2, F_n_1=1,
       F_n_2;
  for (i=3; i<=n; ++i) {
    F_n_2 = F_n_1;
    F_n_1 = F_n;
    F_n = F_n_2 - n*F_n_1;
  }
  return F_n;
}
```


Examples

- **A5 a**) $F(n) = \frac{n^2 + 1\lg(n)}{\sqrt{n+1}}$ F(1), F(8), F(t), F(n-1), F($\sqrt{n}$).
- **b**) $F(n) = (F(n-1))^2 - 3F(n-3)$
- F(t), F(n-1), F(n+3), $F(\lfloor n/2 \rfloor)$
- **A7**) F(0)=2, $F(n) = (F(n-1))^2 - 3F(n-1)$
- Find F(4).
- **A9 a**) F(n) = F(n-1)+1/(n(n+1))
- Verify the solution: $F(n) = \alpha + n/(n+1)$
- **b**) F(n) = 4F(n-1)+12F(n-2)
- Solution: $F(n) = \alpha(-2)^n + \beta\, 6^n$

Examples

- **A11**) Find solutions for
- **a**) F(0)=1, F(n)=n F(n-1)
- **b**) F(0)=0, F(1)=-1,
-     F(n) = -2F(n-1)-F(n-2)
- **A13**) Order, linear, homogeneous?
- **a**) $F(n)=(F(n-1))^2-3F(n-3)$
- **b**) $F(n)=F(n-1)+2F(n-2)+3\ (2)^{n-1}$
- **c**) F(n)=(F(n-1)+F(n-2))/2
- **d**) F(n)=(F(n-1)+1)/2

More Examples

- **A13f**) $\frac{2^n F(n) - 2^{n+1} F(n-1)}{F(n-2) + F(n-3)} = 1 + \lfloor \lg(n!+1) \rfloor$
- **A15**. Given the function F from ex. A3, what's wrong with the following code:

```
for (int k=1; k<=10000; ++k)
  cout << "F(" << k <<")="
       << F(k) << endl;
```

Linear Homogeneous RR with Constant Coefficients

- **Theorem B1 (Superposition)**. Suppose that $F(n)=g(n)$ and $F(n)=h(n)$ are two solutions for
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2)+ ... + c_n(n)F(0)$
- Then every linear combination of g and h is also a solution to the RR:
- $F(n) = \alpha g(n) + \beta h(n)$
- Also true for $\alpha = \beta = 0$

General Solutions

- **Theorem B2**. The solution of the recurrence relation
- $F(n)=c_1 F(n-1), F(n_0)=a$
- is the following:
- $F(n)=\alpha c_1^n$ for all $n \geq n_0$
- **Theorem B3**. The solution for the RR
- $F(n)=c_1 F(n-1)+c_2 F(n-2)$
- is of the form
- $a_1(r_1)^n+a_2(r_2)^n$ or $(a_1+a_2n)(r_1)^n$
- where r_1 and r_2 are the solutions of
- $r^2=c_1r+c_2$

Examples

- **B4**. Find the solution for:
- F(n)=-F(n-1)+2F(n-2), F(0)=3, F(1)=5
- **B5**. Give the form of the solution for
- a) F(n)=5F(n-1)
- b) F(n)=2F(n-1)/3
- **B7**. Solve the following:
- a) F(0)=1, F(n)=2F(n-1)
- d) T(0)=3, T(n)=T(n-1)/2

Examples 2nd Order

- **B5**. Give the form of the solution for
- d) F(n)=F(n-1)+2F(n-2)
- f) F(n)=-3F(n-1)+F(n-2)
- **B9**. Solve the following
- a) F(1)=2, F(2)=0, F(n)=F(n-1)+6F(n-2)
- c) f(0)=0, f(1)=1, f(n)=f(n-1)+f(n-2) (Fibonacci numbers)
- d) A(0)=-3, A(1)=3, A(j)=6A(j-1)-9A(j-2)

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Third Order RR

- The solution for a RR of the form
- $F(n)=c_1 F(n-1) + c_2 F(n-2) + c_3 F(n-3)$
- is of one of the forms
- $F(n)= a_1(r_1)^n+a_2(r_2)^n+a_3(r_3)^n$
- $F(n)= (a_1+a_2n)(r_1)^n+a_3(r_3)^n$
- $F(n)= (a_1+a_2n+a_3n^2)(r_3)^n$
- where r_1, r_2, and r_3 are the solutions for
- $r^3=c_1r^2+c_2r+c_3$

Examples 3rd Order

- F(n) = 3 F(n-1) - 4 F(n-3)
- F(0) = 4
- F(1) = 4
- F(2) = 34

Linear Nonhomogeneous RR

- **Theorem C1**. Suppose F(n)=g(n) is a solution for the relation
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0) + \phi(n)$
- and suppose that h(n) is a solution for
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0)$
- Then $F(n) = g(n) + \alpha\, h(n)$ if also a solution for the first RR.

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Theorem C2

- Suppose F(n)=g(n) and F(n)=h(n) are solutions for the relations
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0) + \phi(n)$
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0) + \psi(n)$
- Then $F(n)=\alpha\, g(n) + \beta\, h(n)$ is a solution for
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0) + \alpha\, \phi(n) + \beta\, \psi(n)$

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Theorem C3

- Suppose F(n)=g(n) and F(n)=h(n) are solutions for the same RR
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0) + \phi(n)$
- The F(n)=g(n)-h(n) is a solution for the homogeneous RR
- $F(n)=c_1(n)F(n-1)+c_2(n)F(n-2) + \dots c_n(n)F(0)$

Example

- $F(n)=3F(n-1)+4(5)^n+7$
- Decompose as
- $F(n)=3F(n-1)+4(5)^n$
- $F(n)=3F(n-1)+7$
- Solutions of the form
- $F(n)=\beta (5)^n$ and $F(n)=\gamma$ (constant)
- Use Theorems C2 and C3.

General Procedure

a) Find the most general solution for the homogeneous part of the RR
b) Find a particular solution for the nonhonogeneous RR
c) Add the two to obtain a general solution for the RR
d) Use the initial values for the sequence to determine the values of all the constants

Polynomial NonH RR First Order

- $F(n)=c_1F(n-1)+P(n)$, where $P(n)$ is a polynomial.
- Solution:
  - if $c1\neq1$, $F(n)=Q(n)$, where $Q(n)$ is a polynomial of the same degree as $P(n)$
  - if $c_1=1$, $F(n)=Q(n)$, where $Q(n)$ is a polynomial of a degree 1 greater than P with the constant term being 0.
- Examples: a) $F(n)=2F(n-1)+2n^3-5n$
- b) $F(n)=F(n-1)+3n$ c) $F(n)=F(n-1)+7$

Second Order

- $F(n)=c_1F(n-1)+c_2F(n-2)+P(n)$, where P is a polynomial.
- Solution: let r_1 and r_2 be the solutions for $r^2=c_1r+c_2$.
- If $r_1 \neq 1 \neq r_2$, then the solution is Q(n), a polynomial of the same degree as P.
- If one of r_1 and r_2 is 1, then the solution is Q(n), a polynomial of degree 1 greater than P, constant term in Q being 0.
- If $r_1=r_2=1$, then the solution is Q(n), a polynomial of degree 2 greater than P, constant term =0 and coefficient of n=0.

Examples

- **C8**
- a) $F(n)=4F(n-1)+5F(n-2)+2n$
- b) $F(n)=-5F(n-1)+6F(n-2)+2n$
- c) $F(n)=2F(n-1)-F(n-2)+2n$
- **C9**. $F(0)=1$, $F(n)=-2F(n-1)+n-1$
- **C11**. To compute $F(n)=\Sigma k^2$ for k from 1 to n, we can express it as
- $F(n) = F(n-1) + n^2$

Theorem C10 (Matching Coefficients)

- Suppose we have an equation of the form
- $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0 = b_k n^k + b_{k-1} n^{k-1} + \ldots + b_1 n + b_0$
- and suppose that the equation is satisfied by an infinite number of values of n. Then $a_0=b_0$, $a_1=b_1$,... $a_k=b_k$.
- Similarly, the equation
- $c_1 q_1^n + c_2 q_2^n + \ldots + c_k q_k^n = d_1 q_1^n + d_2 q_2^n + \ldots + d_k q_k^n$
- is satisfied by an infinite number of values of n iff all coefficient c_i and d_i are respectively equal.

Exponential NonH Linear RR with Constant Coefficients

- **First Order**. $F(n)=c_1 F(n-1)+dq^n$
- If $c_1 \neq q$, $F(n)=\gamma q^n$
- If $c_1=q$, $F(n)=\gamma n q^n$
- **Examples**.
- C13 a) $F(n)=F(n-1)+3\ 2^n$
- b) $F(n)=2F(n-1)+2\ 3^n + 3\ 2^n$

Second Order

- $F(n)=c_1F(n-1)+c_2F(n-2)+dq^n$
- Let r_1 and r_2 be the solutions for $r^2=c_1r+c_2$.
- If $r_1 \neq q \neq r_2$, then $F(n)= \gamma q^n$
- If one of r_1 and r_2 is equal to q, then $F(n)= \gamma n q^n$
- If both $r_1=r_2=q$, then $F(n)= \gamma n^2 q^n$
- *Note1:* only one value of γ will satisfy the recurrence relation. It is determined by replacing the solution in the RR.
- *Note2*: We still need to add to this the general solution to the homogeneous part of the equation. The coefficients of that part are determined by F(0), F(1), etc.

Examples

- **C14 a)** $F(n)=5F(n-1)-6F(n-2) + (-1)^n$
- **b)** $F(n)=F(n-1)+2F(n-2) + (-1)^n$
- **c)** $F(n)=6F(n-1)-9F(n-2)+1/2^n+3^n$
- **C15**. F(1)=1, F(2)=2,
- $F(n)=2F(n-1)+4F(n-2)-5n+2(-1)^{n+1}$

C17

- Which of these are polynomials and of what degree:
- a) n^3+3n^2-2n+1 of n
- b) 3^n-2^n+1 of n
- c) 2^{10} of n
- d) $\sqrt{4n^2-3n+7}$ of n
- e) $(k+1)/2$ of k
- f) $\lfloor (k+1)/2 \rfloor$ of k
- g) p lg m of m (p is constant)
- h) p lg m of p (m is constant)

C19

- a) $F(n)=3F(n-1)+3n^2-2n+1$
- b) $F(n)=F(n-1)+3n^2-2n+1$
- c) $G(k)=G(k-1)-2$
- d) $T(n)=(T(n-1)+n)/3$
- e) $F(n)=3F(n-1)-2F(n-2)-(n+1)/5$
- f) $g(n)=-2g(n-1)+5g(n-2)-4n^2+6$
- g) $F(n)=2F(n-1)-F(n-2)-n^3$
- h) $M(k)=5M(k-1)-6M(k-2)+k^2$

Recurrence Relations in $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$

- **D1**. $F(1)=a$, $F(n) = b\, F(\lfloor n/2 \rfloor)$
- $F(n)=b^2F(\lfloor n/4 \rfloor)=\ldots b^kF(\lfloor n/2^k \rfloor)$
- Iterate until $\lfloor n/2^k \rfloor = 1$, $k= \lfloor \lg n \rfloor$
- $F(n) = a\, b^{\lfloor \lg n \rfloor}$
- **D2**. $F(1)=a$, $F(n)= b\, F(\lfloor n/2 \rfloor) +1$
- Same procedure
- $F(n)= a\, b^{\lfloor \lg n \rfloor}+(b^{\lfloor \lg n \rfloor} -1)/(b-1)$
- $b=1 \Rightarrow F(n)=a+ \lfloor \lg n \rfloor$

More Examples

- **D3**. $F(1)=a$, $F(n)= b\, F(\lfloor n/2 \rfloor) +n$
- Same kind of iteration.
- $F(n)=a\, b^{\lfloor \lg n \rfloor}+\Sigma_{k=1}^{\lfloor \lg n \rfloor}\lfloor n/2^k \rfloor b^k$
- $F(n)=\Theta(n^{\lg b})$
- **D4**. $F(1)=a$, $F(n)=F(\lfloor n/2 \rfloor) +F(\lceil n/2 \rceil)$
- $F(n)=an$
- **D5**. $F(1)=a$, $F(n)=F(\lfloor n/2 \rfloor)+F(\lceil n/2 \rceil)+1$
- $F(n)=(a+1)n-1$

Examples

- **D6**. F(1)=a,
  $F(n) = F(\lfloor n/2 \rfloor) + F(\lceil n/2 \rceil) + n$
- $F(n)=(a+2)n+n\lfloor \lg n \rfloor-2^{\lfloor \lg n \rfloor+1}$
- $F(n) \sim n \lg n$
- **D7**. F(0)=1
  $F(n)=2F(\lfloor n/2 \rfloor) - 7\lfloor n/2 \rfloor + 5n+3$
- $F(n)=\Theta(n \log n)$

Other RRs

1. Consult a reference book.
2. Calculate a good number of terms and try to "guess" the general formula that you can then verify.
3. Manipulate the equation to obtain a more easily solvable one.
4. Transform the equation into an inequality that can be more easily solved.

Examples

- **E1**. $F(0)=a$, $F(n)=F(n-1)+\phi(n)$
- **E2**. $F(1)=1$, $nF(n)=(n-1)F(n-1)+2^n$
- **E3**. $F(1)=1$, $nF(n)=(n+1)F(n-1)+1$
- **E4**. $F(0)=5$, $F(n)=F(\lfloor (n-1)/2 \rfloor)+3$
- **E5**. $F(1)=2$, $F(n)=F(\lfloor n/2 \rfloor) + n \lfloor n/2 \rfloor$
- **E6**. $F(0)=1$, $F(n)=n/(n+2)F(n-1)+ \lfloor n/2 \rfloor$

Theorem E7

- Suppose $F(0), F(1), F(2), \ldots$ satisfy a recurrence inequality of the form $F(n)<c(n)F(n-1)+ \phi(n)$, where $c(n)>0$ for all $n>0$. Let $G(n)$ be defined by $G(0)=F(0)$, $G(n)=c(n)F(n-1)+ \phi(n)$ for $n>0$. Then $F(n)<G(n)$ for any $n>0$.

More Examples

- **E9**. $F(n)=b\,F(\lceil (n+1)/2 \rceil)+\phi(n)$. Prove that it can be transformed into $F(n)=G(n-1)$ where $G(n)=b\,G(\lfloor n/2 \rfloor)+\phi(n+1)$
- **E11**. Solve the problem $F(2)=5$, $F(n)=2F(\lfloor n/2 \rfloor+1)+3\lfloor n/2 \rfloor$, $n\geq 3$ Substitute $F(n)=G(n-1)$.
- **E15**. $F(1)=5$, $F(n)=F(\lfloor n/2 \rfloor)+\sqrt{n}$