C455 Algorithms Analysis

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Overview

- Math preliminaries and review
- Recurrence relations
- Deterministic analysis of algorithms
- Probabilistic analysis of algorithms
- Finite graph algorithms

Definitions

- positive >0 — strictly greater than 0
- negative <0 — strictly lower than 0
- cardinal number ∈ **N** — non-negative whole numbers
- integer number ∈ **Z** — whole numbers
  - {..., -2, -1, 0, 1, 2, ...}
- even — 0, 2, -2, 4, -4, ...
- odd — 1, -1, 3, -3, ...
- rational number ∈ **Q** — fraction of two integer numbers: ½, -2/5, ...
- real number ∈ **R**
- irrational number — $\sqrt{2}$, π, e

Fractions

$$\frac{a(1+x)}{a(y-2)} = \frac{1+x}{y-2}$$

$$\frac{a}{b} \cdot \frac{x}{y} = \frac{ax}{by}$$

$$\frac{a}{b} + \frac{x}{y} = \frac{ay+bx}{by}$$

$$\frac{x+y}{x+z} \neq \frac{y}{z}$$

$$\frac{a}{b} + \frac{x}{y} \neq \frac{a+x}{b+y}$$

Square Root

- If $x \geq 0$, the square root of x is a number denoted by $\sqrt{x}$ such that

$$\left(\sqrt{x}\right)^2 = x$$

- Generally, the root of n degree of x is a number such that

$$\left(\sqrt[n]{x}\right)^n = x$$

$$\sqrt{xy} = \sqrt{x}\sqrt{y} \qquad \sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$$

Laws of Exponents

- $(x\ y\ z)^p = x^p\ y^p\ z^p$
- $x^{p+q} = x^p x^q$
- $x^{pq} = (x^p)^q = (x^q)^p$
- $x^{-p} = 1/x^p$

- $(x+y+z)^p \neq x^p + y^p + z^p$

Floor and Ceiling

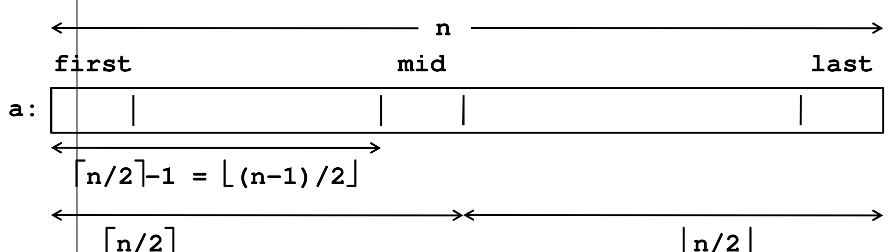
- Def. The *floor* of a real number *x*, denoted by $\lfloor x \rfloor$, is the largest integer that is lower or equal than *x*.
- $\lfloor 2.5 \rfloor = 2$, $\lfloor 0.01 \rfloor = 0$, $\lfloor -1.3 \rfloor = -2$
- Def. The *ceiling* of a real number *x*, denoted by $\lceil x \rceil$, is the smallest integer that is greater or equal than *x*.
- $\lceil 2.5 \rceil = 3$, $\lceil 0.01 \rceil = 1$, $\lceil -1.3 \rceil = -1$

Theorem A3

- Let $x, y \in \mathbf{R}$, $n, m, k \in \mathbf{Z}$.
- a) $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$,
- b) $\lceil x \rceil - 1 < x \leq \lceil x \rceil$
- c) if $n \leq x < n+1$, then $n = \lfloor x \rfloor$
- d) if $n-1 < x \leq n$, then $n = \lceil x \rceil$
- e) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$, $\lceil x+n \rceil = \lceil x \rceil + n$
- f) $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$
- g) $\lfloor (n+1)/2 \rfloor = \lceil n/2 \rceil$
- h) $\lfloor \lfloor y \rfloor /k \rfloor = \lfloor y/k \rfloor$, $\lceil \lceil y \rceil /k \rceil = \lceil y/k \rceil$, for $k>0$
- i) if $n, m>0$, $\lfloor n/m \rfloor = \lceil (n+1)/m \rceil - 1$
- j) if $n, m>0$, $(n-m+1)/m \leq \lfloor n/m \rfloor \leq n/m$

Theorem A4

- Consider a subarray a[first..last] of an array a, where 0 $\leq$ first $\leq$ last. Let n be the length of the subarray, and mid= $\lfloor$(first+last)/2$\rfloor$. Then
- a) n = last – first + 1;
- b) the length of the subarray a[first..mid] is $\lceil$n/2$\rceil$;
- c) the length of the subarray a[first..mid-1] is $\lceil$n/2$\rceil$-1 = $\lfloor$(n-1)/2$\rfloor$;
- d) the length of the subarray a[mid+1..last] is $\lfloor$n/2$\rfloor$.



C455 Algorithms Analysis

Examples

- $0 \le x - \lfloor x \rfloor < 1$
- $\lfloor 2^m + \log_2 m \rfloor = 2^m + \lfloor \log_2 m \rfloor$
- $n - 2 * (n/2)$, $n=15, 22$
- $\lceil x \rceil + \lceil y \rceil \le \lceil x+y \rceil + 1$

```
for (int k=1; (3*k)+1 <= n; ++k)
   cout << k << endl;
```


Logarithms

- **Def**. For a given real number $b>1$, the logarithm base b of any real number x is defined as the power of b that produces x and is denoted by $\log_b x$.
- **Theorem B2**.
- a) $b^{\log_b x} = x$.
- b) $\log_b (b^y) = y$
- for any x, y, b.

Theorem B3

- For any fixed real number b>1,
- a) $\log_b(s\,t) = \log_b s + \log_b t$
- b) $\log_b(s/t) = \log_b s - \log_b t$
- c) $\log_b(t^p) = p \log_b t$
- **Example**
- $\log_{10}1000$, $\log_{10}100$, $\log_b 1$, $\log_{10}0.01$
- **Natural logarithm** in base e~2.71828..

$$\log_e x = \ln x = \int_1^x \frac{1}{t}\,dt$$

Theorem B6

- For any a, b $\in$ **R**, a>1, b>1,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

- Examples:
- $\log_a x = \log_a b \log_b x$
- $\log_a b = 1/\log_b a$
- $\ln x \sim 2.303 \log_{10} x$,
  2.303 ~ ln 10
- $\log_2 x = \lg x \sim 3.32 \log_{10} x$,
  3.32 ~ lg 10

Theorem B8

- Let $n \in \mathbf{N}$, $n>0$.
- a) $2^{\lfloor \lg n \rfloor} \le n < 2^{\lfloor \lg n \rfloor + 1}$
- b) $\lfloor n / 2^{\lfloor \lg n \rfloor} \rfloor = 1$, $\lfloor n / (2^{\lfloor \lg n \rfloor} + 1) \rfloor = 0$
- c) $\lceil \lg(n+1) \rceil = 1 + \lfloor \lg n \rfloor$
- d) $\lfloor \lg \lfloor n/2 \rfloor \rfloor = -1 + \lfloor \lg n \rfloor$
- e) $\lceil \lg \lceil n/2 \rceil \rceil = -1 + \lceil \lg n \rceil$
- f) $\lfloor \lg \lceil n/2 \rceil \rfloor = -1 + \lfloor \lg (n+1) \rfloor$
- g) $\lceil \lg \lfloor n/2 \rfloor \rceil = -1 + \lceil \lg (n-1) \rceil$

Examples

- **Theorem B10**. $x^{\log_b y} = y^{\log_b x}$.
- $\ln e^{7x}$, $\ln x^7$, $\ln(7x)$, $e^{x + \ln 7}$,
- $\lg 2^{-n}$, $\lg (n/32)$, $2^{3 \lg 3}$, $\lg(\sqrt{2})$
- $e^{n \ln x}$, $e^{\ln n - \ln m}$, $\lg 32^{20}$, $\lfloor \lg 1500 \rfloor$

```
int k=1;
for (int p=2; p<=n; p*=2) {
  cout << k << endl;
  ++k;
}
```

Sums and Summation Notation

- To sum the values of a function for a list of integer in a range between m and n, we use the Σ notation:

$$\sum_{k=m}^{n} \mathrm{f}(k)$$

- The result can be obtained by the following code:

```
sum = 0;
for (int k=m; k<=n; ++k)
  sum += f(k);
```


Examples

$$\sum_{k=2}^{5} \frac{k}{k+1}$$

$$\sum_{k=0}^{7} k2^{k-1}$$

$$\sum_{k=4}^{4} \left\lfloor \frac{15}{k} \right\rfloor$$

$$\sum_{k=10}^{25} 3$$

$$\sum_{j=1}^{3} \left(\sum_{k=1}^{j} (j-k)^2 \right)$$

$$\sum_{k=2}^{7} k(k+1) = \sum_{i=2}^{7} i(i+1) = \sum_{\alpha=2}^{7} \alpha(\alpha+1)$$

Properties

- The sum is a linear operator

$$\sum_{k=m}^{n} c\,\mathrm{f}(k) = c\sum_{k=m}^{n}\mathrm{f}(k)$$

$$\sum_{k=m}^{n}\left(\mathrm{f}(k)+\mathrm{g}(k)\right) = \sum_{k=m}^{n}\mathrm{f}(k)+\sum_{k=m}^{n}\mathrm{g}(k)$$

$$\sum_{k=m}^{n}\left(\sum_{j=p}^{q}\mathrm{f}(j,k)\right) = \sum_{j=p}^{q}\left(\sum_{k=m}^{n}\mathrm{f}(j,k)\right)$$

Well-Known Sums

- Triangular sums

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=m}^{n} k = (n-m+1)\frac{m+n}{2}$$

- Finite geometric sums

$$\sum_{k=0}^{n} t^k = \frac{t^{n+1}-1}{t-1}$$

- Harmonic sums

$$\sum_{k=1}^{n}\frac{1}{k}$$

Harmonic Sum Approximation

- **Theorem C5**

$$\int_m^{n+1} f(x)\,dx < \sum_{k=m}^{n} f(k) < f(m) + \int_m^{n} f(x)\,dx \quad \text{if f is decreasing}$$

$$f(m) + \int_m^{n} f(x)\,dx < \sum_{k=m}^{n} f(k) < \int_m^{n+1} f(x)\,dx \quad \text{if f is increasing}$$

- Example

$$\int_1^{n+1} \frac{1}{x}dx < \sum_{k=1}^{n} \frac{1}{k} < \frac{1}{1} + \int_1^{n} \frac{1}{x}dx$$

$$\ln(n+1) < H(n) < 1 + \ln(n)$$

$$H(n) \approx 0.5772 + \ln(n) \quad \textit{Euler}$$

Theorems

- **C8**

$$1 + 2t + 3t^2 + \ldots + nt^{n-1} = \frac{1 + nt^{n+1} - (n+1)t^n}{(t-1)^2}$$

- **C9**

$$\sum_{k=2}^{n} \lceil \lg k \rceil = n\lfloor \lg n \rfloor + (n+1) - 2^{\lfloor \lg n \rfloor + 1}$$

$$\sum_{k=2}^{n} \lfloor \lg k \rfloor = (n+1)\lfloor \lg(n+1) \rfloor + 2 - 2^{\lfloor \lg(n+1) \rfloor + 1}$$

- **Application**. The number of digits in the representation of n base b is

$$1 + \lfloor \log_b n \rfloor = \lceil \log_b (n+1) \rceil$$

Triangular Sums

- 1000+999+998+...+1
- 100+101+102+...+349+350
- 5+10+15+...+745+750
- 5+8+11+...+98
- 1+2+3+...+(2n-1)+2n

```
for (int k=2; k<=50; ++k)
  for (int j=1; j<k; ++j)
    cout << "hello" << endl;
```


Geometric and Harmonic Sums

- $1+4+16+64+...+4^7$
- $1+1/3+1/9+1/27+...+1/3^{10}$
- $1+x+x^2+...+x^{n-1}$
- $1+x^3+x^6+...+x^{3n}$

$$\sum_{k=1}^{n} \frac{1}{k^2} \qquad \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \qquad \sum_{k=1}^{2^n} \frac{1}{k} \approx n \ln 2$$

Mathematical Induction

- **Principle of Weak Mathematical Induction**: Let $S(n)$ denote any predicate in which the variable n has as its universe the set of all positive integers. Then
- $[\ S(1) \wedge \forall n\ (\ S(n) \rightarrow S(n+1)\ )\ ] \rightarrow \forall n\ S(n)$.
- (i) Prove that $S(1)$ is true; this is called "establishing (or verifying) the ***base case***."
- (ii) Prove that $\forall n\ (\ S(n) \rightarrow S(n+1)\ )$; this is called the ***inductive step*** of the proof.

Mathematical Induction

- Generalized weak induction:
- $[\ S(n_0) \wedge \forall\ n\ (\ (\ n \geq n_0 \wedge S(n)\ ) \rightarrow S(n+1)\ )\ ] \rightarrow \forall\ n\ S(n)$
- Alternate form:
- $[\ S(n_0) \wedge S(n_0+1) \wedge \forall\ n\ (\ (\ n > n_0 \wedge S(n) \wedge S(n-1)\ ) \rightarrow S(n+1)\ )\ ] \rightarrow \forall\ n\ S(n)$
- The principle of strong induction:
- $[\ S(n_0) \wedge (\forall n\ (\forall k, n_0 \leq k \leq n, S(k)\ ) \rightarrow S(n+1)\ )\ ] \rightarrow \forall\ n\ S(n)$

Factorials

- The factorial of a non-negative integer number n, denoted by n! is defined as
- $0! = 1$, $1!=1$, $n!=1\cdot 2\cdot 3\ldots (n-1)n$
- $n! = (n-1)! \cdot n$
- $(2n)! > 2^n\, n!$
- **Theorem D3**:
- $n \ln n - n+1 < \ln(n!) < (n+1)\ln(n+1)-(n+1)+1$
- **Corolary**: $e(n/e)^n < n! < e((n+1)/e)^{n+1}$

Theorem D5

- Stirling's Formula: $$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

$$\lim_{n\to\infty} \frac{n!}{\left(n/e\right)^n \sqrt{2\pi n}} = 1$$

- Examples: 13!/(4! 9!) 20!/(15! 5!)
- $(n^2)!$ $(2^n)!$
- (n+1)!/n! n!/(n+2)! (n-1)!/(n+1)!

Proof of Stirling's Formula

- Involves defining a continuous function such that

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$$

- $\Gamma(n+1) = n!$ if $n \in$ **N**, $\Gamma(x+1) = x\,\Gamma(x)$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-t^2}\, dt = \sqrt{\pi}, \quad u = t^2$$

$$\Gamma(x+1) = \int_0^\infty u^x e^{-u}\, du = x^{x+1}\int_0^\infty t^x e^{-xt}\, dt, \quad u = xt$$

$$\frac{\Gamma(x+1)}{x(x^x)} = \int_0^\infty (te^{-t})^x\, dt$$

- From there the interval [0, ∞] is divided in 3, [0,a], [a,b], [b, ∞] where 0 < a < 1 < b and so on. Another substitution used: $t = s\sqrt{c}\,x$

Little "oh" Notation

- **Def**. Let f(n) and g(n) be positive functions defined on positive integers. We say that f(n) is o(g(n)) as n->∞ if

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

- **Examples**
- $n^2 = o(n^3)$, f(n)=o(1)? lg n = o(n)
- 3n=o(10n)? $\lg(n^2) = o(\lg(n^3))$?
- **Theorem E3**. If f(n) = o(c g(n)), then f(n)=o(g(n)).

Theorem E4

- Let $\lim_{n\to\infty} F(n)=L$ and $\lim_{n\to\infty} G(n)=M$
- a) $\lim_{n\to\infty} c\,F(n)=c\,L$.
- b) $\lim_{n\to\infty}[F(n)+G(n)]=L+M$
- c) $\lim_{n\to\infty}[F(n)\,G(n)]=L\,M$
- d) $\lim_{n\to\infty}[F(n)/G(n)]=L/M$ if $M\neq 0$
- e) $\lim_{n\to\infty}[F(n)]^p=L^p$
- f) If $\lim_{n\to\infty} F(n)=\infty$, and $c>0$, then $\lim_{n\to\infty}[cF(n) + b]= \infty$

Theorem E4

- g) If $L=0$, $c>0$, and $F(n)>0$, then $\lim_{n\to\infty}[c/F(n)] = \infty$
- h) If $L=M=0$ or ∞, then $\lim_{n\to\infty}[F(n)+G(n)]=\lim_{n\to\infty}[F(n)G(n)] = L$ but not $\lim_{n\to\infty}[F(n)/G(n)]$
- i) If $F(n)>0$, then $L=0 \Leftrightarrow \lim_{n\to\infty}[\ln(F(n))]=-\infty$
- j) if $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$
- k) if $f(n)\leq g(n)\leq h(n)$ and $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} h(n) = L$, then $\lim_{n\to\infty} g(n)=L$

Examples

- $\lim_{n\to\infty}\sqrt{F(n)} = \sqrt{\lim_{n\to\infty}F(n)}$
- $\lim_{n\to\infty}(3/\lg n) = 0$
- $\lim_{n\to\infty}\lfloor \lg 7 \rfloor (2/3)^n = 0$
- $\lim_{n\to\infty}(2/(9/10)^n) = \infty$
- $\lim_{n\to\infty}\ln(2/n+3/n^2) = -\infty$

Examples

- $\sqrt{n}=o(n)$, $\lg n = o(\sqrt{n})$, $n^{100}=o(2^n)$
- $10^n=o(n!)$
- n^2 2^n $\sqrt{n}$ $\sqrt{n}\lg(n)$ $n!$ 4^n $\ln n$ $10n$ $n\lg(n)$
- **Theorem E8**. If $f(n) = o(g(n))$, then for any $\varepsilon>0$ (no matter how small) $f(n) < \varepsilon\, g(n)$ for n large enough.
- $\lg(2^n+n) = n+o(1)$

More Examples

- Which of these are true?
- $0.1\ n^3 = o(n^3)$
- $2^n=o(3^n)$ $2^n=o(2^{n+1})$
- $6\ (\ln n)^3=o(n)$ $n^2 \lg n = o(n^3)$
- $\lg(n^3)=o((1.5)^n)$ $(n \lg n)^2=o(n^2 \sqrt{n})$

Big O, Big Ω, Big Θ

- Let f(n) and g(n) be positive functions defined on positive integers. We say that.
- $f(n)=O(g(n))$ if $\exists c>0,\ n_0\geq 0$ such that $f(n) \leq c\ g(n)$ for $n\geq n_0$
- $f(n)=\Omega(g(n))$ if $\exists c>0,\ n_0\geq 0$ such that $f(n) \geq c\ g(n)$ for $n\geq n_0$
- $f(n)=\Theta(g(n))$ if $f(n)=O(g(n))$ and $f(n)=\Omega(g(n))$

Examples O

- $3n+7 = O(n)$ $7n^2-6n = O(n^2)$
- $\lg(7n^2+4)=O(\lg n)$ $\ln(n!)=O(n \ln n)$
- $4 \lg n = O(n)$
- **Theorem F3**. If $f(n) = o(g(n))$, then $f(n) = O(g(n))$ but $g(n) \neq O(f(n))$.
- $5n^3=O(n^3)$ $100n^2=O(n^4)$
- $\lg n^2 = O(\lg n)$ $(n^2+7n-10)^3=O(n^6)$
- $f(n)=O(1)$

Examples Ω

- $3n+7 = \Omega(n)$ $7n^2-6n = \Omega(n^2)$
- $0.5n–6 \lg n=\Omega(n)$ $\ln(n!)=\Omega(n \log n)$
- If $f(n) = \Omega(g(n))$, then $g(n)=O(f(n))$.
- $f(n)=\Omega(1)$
- If $f(n)=o(g(n))$ then $g(n)=\Omega(f(n))$ but $f(n) \neq \Omega(g(n))$.
- $5n^3= \Omega(n^3)$ $100n^2= \Omega(n^4)$
- $\sqrt{n} = \Omega((\lg n)^3)$ $\lg(n^2-n) = \Omega(\log n)$

Examples Θ

$$\log_b(n!) = \sum_{k=1}^{n} \log_b k = \Theta(n \log n)$$

$$\sum_{k=1}^{n} k^p = \Theta(n^{p+1})$$

- **Theorem F9**. Suppose f(n) and A(n) are positive functions such that
- $f(n) = bA(n) + b_1h_1(n) + ... + b_kh_k(n)$
- where $b>0$, $b_i \neq 0$, $k>0$, and for every i,
- $h_i(n) = o(A(n))$
- Then $f(n) = \Theta(A(n))$.
- Examples. $f(n)=3lg(n)-7\sqrt{n}+1/2n\ lg(n) -10n$
- $f(n)=2^n-7n^2+4n\ lg(n)+n!$

Theorem F11

- Let f(n), g(n), A(n), B(n) be positive functions defined on positive integers.
- a) Transitivity: If f(n)=O(A(n)) and A(n)=O(B(n)), then f(n)=O(B(n))
- b) If f(n)=O(A(n)), then f(n)g(n)=O(A(n)g(n))
- c) f(n)=O(A(n)) and g(n)=O(A(n)), then $\alpha f(n)+\beta g(n)=O(A(n))$, a,b>0
- d)If f(n)=O(A(n)) and g(n)=O(B(n)), then f(n)+g(n)=O(A(n)+B(n)) and f(n)g(n)=O(A(n)B(n))
- If f(k)=O(A(k)), then $\Sigma f(k)= O(\Sigma(A(k))$

Theorem F11

- e) If $f(n)=\Theta(A(n))$ and $g(n)= \Theta(B(n))$, then $f(n)/g(n)= \Theta(A(n)/B(n))$
  (not true for O and Ω)
- f) If $f(n)\leq g(n)$ for n large enough and $g(n)=O(A(n))$, then $f(n)=O(A(n))$
- If $f(n)\geq g(n)$ for n large enough and $g(n)=\Omega(A(n))$, then $f(n)=\Omega(A(n))$.
- If $g(n)<f(n)<h(n)$ for n large enough and $g(n)=\Theta(A(n))$ and $h(n)=\Theta(A(n))$, then $f(n)= \Theta(A(n))$
- g) If $f(n)=\phi(n)+\varphi_1(n)+...+\varphi_k(n)$ and $\phi(n)=O(A(n))$ and $\varphi_i(n)=O(h_i(n))$ with $h_i(n)=o(A(n))$, then $f(n)=O(A(n))$

More Examples

- $(n+1)\lg(n+1)=\Theta(n \log n)$
- **Def**. $f(n)=g(n)+O(A(n))$ iff $|f(n)-g(n)|=O(A(n))$
  (same for Ω and Θ)
- $f(n)=(n^5+2)/(n^2+5)$, $f(n)=n^3+\Theta(n)$
- $5n^3 = \Theta(n^3)$ $100n^2 = \Theta(n^4)$?
- $f(n) = \Theta(1)$ $4n^2+5n-8 = \Theta(n^2)$
- $\lg(1)+\lg(2)+...+\lg(n)=O(\lg(n))$??

Number of Iterations

```
for (int k=1; k<=n; ++k)
  for (int j=1; j<=k; ++j)
    for (int i=1; i<=j; ++i)
      cout << "hello" << endl;
```


Asymptotic Functions

- **Def**. Let f(n) and A(n) be positive functions defined for all n>0. We say that f(n) is asymptotic to A(n) as $n \to \infty$ if $\lim_{n\to\infty} f(n)/A(n)=1$. Denoted by $f(n) \sim A(n)$.
- **Examples**. $3n^2+5n-12 \sim 3n^2$
- $((n+1)2^n)/(2^{n+1}+1) \sim n/2$
- $H(n) \sim \ln(n)$
- $n! \sim (n/e)^n\sqrt{2\pi n}$

Theorem G6

- Suppose
- $f(n)=A(n)+c_1g_1(n)+...+c_kg_k(n)$
- and $g_i(n)=o(A(n))$
- Then $f(n) \sim A(n)$.
- Example.
- $f(n)=3lg(n)-6n+5n^2+7n\ lg(n)$

Theorem G8

- Suppose $f(n) \sim A(n)$ and $g(n) \sim B(n)$.
- a) $f(n)g(n) \sim A(n)B(n)$
- b) $f(n)/g(n) \sim A(n)/B(n)$
- c) For any $c \neq 0$, $c\ f(n) \sim c\ A(n)$
- d) If $A(n), B(n)>0$, then for any $\alpha>0$, $\beta>0$, $\alpha\ f(n)+\beta\ g(n) \sim \alpha\ A(n)+\beta\ B(n)$
- e) If $A(n) \equiv B(n)$, then for any $\alpha>0$, $\beta>0$, $\alpha\ f(n)+\beta\ g(n) \sim (\alpha+\beta)\ A(n)$
- f) If $f(n) \sim A(n)$ and $A(n) \sim D(n)$, then $f(n) \sim D(n)$.

More Theorems

- **Theorem G9.** If f(n)~A(n), then f(n)=Θ(A(n)).
- **Theorem G10.** If $\lim_{n\to\infty}$f(n)/A(n)
- a) exists and is finite, then f(n)= Θ(A(n)).
- b) exists and is 0, then f(n)=O(A(n)) but not Ω(A(n)), nor Θ(A(n)).
- c) exists and is infinite, then f(n)= Ω(A(n)), but not O(A(n)) , nor Θ(A(n)).
- d) does not exist, no conclusion.